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The S-FFT calculation of Collins formula and its application in digital holography

J.C. Li[a], J. Zhu, and Z.J. Peng

Kunming University of Science and Technology, Kunming 650093, P.R. China

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Abstract. In this paper the single two dimensional fast Fourier transform (S-FFT) calculation of Collins' formula is studied based on Nyquist sampling theorem. The conditions that sampling theorem should follow are deduced, and the inverse calculation of Collins' formula is also applied to the reconstruction of light wave field of object in digital holography. The investigation indicates that inverse calculation of Collins' formula can be used conveniently in digital holography study on paraxial optical system. The sampling conditions derived from this paper are helpful to the design of digital holographic optical system.

PACS. 42.40.Lx Diffraction efficiency, resolution, and other hologram characteristics

1 Introduction

It is a widely adopted method to reconstruct the optical wave field of object plane by using Fresnel diffraction integral in the application research of digital holography [1]. And the calculation of Fresnel diffraction integral is completed by using fast Fourier transform (FFT) [2,3]. However, theoretical analysis has already pointed out that amplitude and phase of diffraction field can only be calculated more accurately when specific conditions are satisfied. For example, when the light wavelength is λ, the diffraction distance is d, and the sampling number is N, the specific condition is that the calculation width of diffraction field is $\sqrt{\lambda d N}$ [2,3]. In actual study, the size of CCD screen is always fixed and invariable. In order to achieve the accurate reconstruction of amplitude and phase of object light field, the sampling data of CCD must be done by zero-filling operation, and a bigger array should be established to complete calculations when the projection size of the object measured is larger than the size of CCD screen. In addition, for larger objects, there is generally a larger diffraction distance. When d is large, the object light arriving at CCD is usually weak, and the signal noise ratio is also low. If object light energy can be converged effectively on the CCD screen through an optical system, the measured signal-noise ratio can be increased. However, this will involve the reconstruction when the object light wave arrives at CCD through an optical system, so it is difficult to reconstruct object light wave field by using Fresnel diffraction integral.

Because of convenient use of Collins' formula in the diffraction research on paraxial optical system, in this paper, we will discuss Collins' formula and S-FFT (single two dimensional fast Fourier transform) algorithm of its inverse calculation based on Nyquist sampling theorem. The study results indicate that the amplitude and phase of incident plane light wave field can be reconstructed well with the help of inverse calculation of Collins' formula under certain circumstances. And the formula can also be applied conveniently in the digital holography research on paraxial optical system. Examples of wavefront reconstruction based on our study results will be given.

2 Collins' formula and its inverse calculation

Supposing the axisymmetrical paraxial optical system can be described as a matrix of order 2, i.e. $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and the coordinates of incident plane and emergent plane are defined as x_0y_0 and xy respectively, Collins established the relation between the optical wave field on the incidence plane and the optical wave field on the emergence plane, which can be described as equation (1) [4]:

$$U(x,y) = \frac{\exp(jkd)}{j\lambda B} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U_0(x_0,y_0) \times \exp\left\{\frac{jk}{2B}\left[A\left(x_0^2+y_0^2\right) + D\left(x^2+y^2\right) - 2\left(xx_0+yy_0\right)\right]\right\} \mathrm{d}x_0\mathrm{d}y_0 \quad (1)$$

where d is optical length along the axis of $ABCD$ optical system. $k = 2\pi/\lambda$, and λ is optical wavelength.

[a] e-mail: jcli@vip.163.com

It can be proved theoretically that there is an inverse calculation expression of Collins' formula, which is expressed as equation (2):

$$U_0(x_0,y_0)=\frac{\exp(-jkd)}{-j\lambda B}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}U(x,y)\times\exp\left\{-\frac{jk}{2B}\left[D\left(x^2+y^2\right)+A\left(x_0^2+y_0^2\right)-2\left(x_0x+y_0y\right)\right]\right\}\mathrm{d}x\mathrm{d}y. \quad (2)$$

Therefore, equations (1) and (2) comprise the calculation relation between the optical wave field on the incidence plane and the optical wave field on the emergence plane of the axisymmetrical paraxial optical system.

It is not difficult to find that if we define xy-plane as the CCD detector plane in the digital holography testing, the optical wave field of objective plane $U(x_0,y_0)$ can be reconstructed with the help of equation (2) once $U(x,y)$ can be obtained from the hologram detected by CCD. Thus, it is very important to study the numeric calculation method that satisfies the sampling theorem of equations (1) and (2).

3 The S-FFT calculation of Collins formula

Equation (1) can be expressed by Fourier transform:

$$U(x,y)=\frac{\exp(ikd)}{i\lambda B}\exp\left[\frac{ik}{2B}D\left(x^2+y^2\right)\right]\times F\left\{U_0(x_0,y_0)\exp\left[\frac{ik}{2B}A\left(x_0^2+y_0^2\right)\right]\right\}_{f_x=\frac{x}{\lambda B},f_y=\frac{y}{\lambda B}}. \quad (3)$$

The equation shows that the computation process of Collins diffraction integral may be regarded as a Fourier transformation of a product of input signal and quadratic phase factor, but the result of Fourier transform needs to be multiplied by another quadratic phase factor.

If the spatial range widths of the optical wave field of incident plane and that of the optical wave field of emergent plane are respectively. ΔL_0 and ΔL in FFT calculation, and the sampling number is $N\times N$, according to discrete Fourier transform theory, the frequency range width is $N/\Delta L_0$ after discrete transformation. Thus, we get:

$$\frac{\Delta L}{\lambda B}=\frac{N}{\Delta L_0},\quad \text{or}\quad \Delta L_0\Delta L=\lambda BN. \quad (4)$$

As $\frac{\Delta L}{N}=\frac{1}{\Delta L_0}\lambda B$ is the spatial range sampling unit of discrete transformation calculation result, the expression of equation (3) after sampling operation can be described as:

$$U\left(p\frac{\lambda B}{\Delta L_0},q\frac{\lambda B}{\Delta L_0}\right)=\frac{\exp(ikd)}{i\lambda B}\exp\left[i\pi\frac{\lambda BD}{\Delta L_0^2}\left(p^2+q^2\right)\right]\times FFT\left\{U_0\left(m\frac{\Delta L_0}{N},n\frac{\Delta L_0}{N}\right)\exp\left[i\pi\frac{A\Delta L_0^2}{\lambda BN^2}\left(m^2+n^2\right)\right]\right\} \quad (5)$$

$$(p,q,m,n=-N/2,-N/2+1,\cdots,N/2-1).$$

However, only calculations that satisfy sampling theorem will not cause frequency spectrum superposition, and more accurate calculation results can be obtained. Analysis of equation (3) indicates that the transformed function is the result of object function multiplied by exponent phase factor. The Fourier transformation of exponent phase factor

$$\exp\left[\frac{ik}{2B}A\left(x_0^2+y_0^2\right)\right]$$

is

$$\frac{\lambda B}{iA}\exp\left(-i\lambda BA\pi\left(\left(\frac{x}{\lambda B}\right)^2+\left(\frac{y}{\lambda B}\right)^2\right)\right).$$

It is a non-band-limited function, which has value in entire frequency domain. According to frequency domain convolution theory, whether object function is a non-band-limited function or not, the convolution result is always a non-band-limited function. Therefore, it is impossible to make the DFT calculation of equation (3) rigidly meet Nyquist sampling theorem. However, Nyquist sampling theorem can be described formally that the reciprocal of spatial sample spacing is larger than or equal to double times of function's maximal frequency spectrum. That is to say, there are at least two sampling points in the space period corresponding to the maximal frequency spectrum. In actual diffraction calculation, computations are usually done to meet sampling theorem approximately based on the following analysis [2,3].

Generally, the spatial change ratio of object function corresponding to exponent phase factor is not high. And the DFT sampling of equation (4) can be approximately regarded as only related to the sampling of exponent phase factor. If there are at least two sampling points in a period 2π of exponent phase factor in the domain defined by ΔL_0, the DFT calculation will be considered to meet sampling theorem approximately. The highest spatial frequency points of quadratic phase factor correspond to the sampling values when both m and n equal $\pm N/2$ in equations (4). Therefore, when solving the inequation at the edge of range:

$$\left|\frac{\partial}{\partial m}\left(\pi\frac{A\Delta L_0^2}{\lambda BN^2}\left(m^2+n^2\right)\right)\Big|_{m,n=N/2}\right|\leqslant\pi.$$

We obtain:

$$|B|\geqslant\frac{|A|\,\Delta L_0^2}{\lambda N}. \quad (6)$$

Equations (6) can be regarded as the condition of S-FFT transform method to obtain diffraction field intensity distribution. In order to make the computed results meet sampling theorem, the quadratic phase factor's sampling in front of DFT of equations (5) also should meet following inequation:

$$\left|\frac{\partial}{\partial p}\pi\frac{\lambda BD}{\Delta L_0^2}\left(p^2+q^2\right)\Big|_{p,q=N/2}\right|\leqslant\pi.$$

Resolving the above inequality, we obtain:

$$|B|\leqslant\frac{\Delta L_0^2}{N\lambda\,|D|}. \quad (7)$$

According to equations (6) and (7), we will obtain:

$$|A| \leqslant \frac{|B|\,\lambda N}{\Delta L_0^2} \leqslant \frac{1}{|D|}. \tag{8}$$

Equation (8) gives the relationship among each element, when the S-FFT calculation of Collins' formula satisfies Nyquist's sampling condition approximately. It also indicates that if system parameters meet $|A| > 1/|D|$, there will be no solution which meets the sampling request for both amplitude and phase at the same time.

Now we will discuss spatial range width of the diffraction field's when S-FFT calculation is used. According to $\Delta L_0 \Delta L = \lambda BN$ in equation (4), when the width of input plane ΔL_0 of $ABCD$ system is defined, and there are limited sampling number N, and B is close to zero, the sampling range width of the calculation ΔL approaches to zero. On the contrary, the output diffraction field scope ΔL will expand linearly with the increase of B. According to the fact that when B approaches to zero, the output plane approaches to the object plane or the image plane, it will be difficult to calculate the diffraction field of near field or that of approaching image plane of $ABCD$ system with S-FFT algorithm.

4 S-FFT calculation of the inverse calculation expression of Collins' formula

The inverse calculation expression of Collins' formula (see Eq. (2)) can be described by inverse Fourier transform as:

$$U_0(x_0, y_0) = \frac{\exp(-ikd)}{-i\lambda B} \exp\left[-\frac{ik}{2B} A\left(x_0^2 + y_0^2\right)\right] \times F^{-1}\left\{U(x,y)\exp\left[-\frac{ik}{2B} D\left(x^2+y^2\right)\right]\right\}_{f_x = \frac{x_0}{\lambda B}, f_y = \frac{y_0}{\lambda B}}. \tag{9}$$

It is obvious that the computation process of inverse calculation of Collins' formula may be regarded as an inverse Fourier transformation of a product of input plane optical wave field and quadratic phase factor. But the result of inverse transform should be multiplied by another quadratic phase factor.

Suppose spatial range widths of the incident plane optical wave field and that of the emergent plane optical wave field are respectively. ΔL_0 and ΔL when inverse fast Fourier transform (IFFT) is used, and the sampling number is $N \times N$, according to equation (4), we can obtain:

$$\Delta L_0 = \frac{\lambda |B| N}{\Delta L}. \tag{10}$$

As $\frac{\Delta L_0}{N} = \frac{\lambda |B|}{\Delta L}$ is the spatial range sampling unit of IFFT calculation result, the S-FFT calculation of inverse calculation of Collins formula can be expressed as equation (11):

$$U_0\left(m\frac{\lambda B}{\Delta L}, n\frac{\lambda B}{\Delta L}\right) = \frac{\exp(-ikd)}{-i\lambda B}\exp\left[-i\pi\frac{\lambda BA}{\Delta L^2}\left(m^2+n^2\right)\right] \times IFFT\left\{U\left(p\frac{\Delta L}{N}, q\frac{\Delta L}{N}\right)\exp\left[-i\pi\frac{D\Delta L^2}{\lambda BN^2}\left(p^2+q^2\right)\right]\right\} \tag{11}$$

$$(m, n, p, q = -N/2, -N/2+1, \cdots, N/2-1).$$

According to the discussion on equation (6), we can obtain the sampling condition of amplitude about equation (11):

$$|B| \geqslant \frac{|D|\,\Delta L^2}{\lambda N}. \tag{12}$$

According to the discussion on equation (7), we can obtain the condition of the front phase factor about equation (11) when it meets sampling theorem approximately:

$$|B| \leqslant \frac{\Delta L^2}{N\lambda\,|A|}. \tag{13}$$

In order to make equation (11) meet sampling theorem approximately, we resolve the equations that are made up of equations (6) and (7), and then we will obtain:

$$|D| \leqslant \frac{|B|\,\lambda N}{\Delta L^2} \leqslant \frac{1}{|A|}. \tag{14}$$

It is obvious that if system parameters meet $|A| > 1/|D|$, then there will be no resolution which meets the sampling request for both amplitude and phase at the same time.

At last we will consider the problem of how to obtain the spatial range width of incident plane diffraction field of $ABCD$ system with using inverse calculation of S-FFT. According to equation (12), when the output plane width ΔL of $ABCD$ system is defined, the inequation can only be satisfied when the sampling number N is very large if B approaches to zero. According to the fact that if B approaches to zero, the output plane approaches to the incidence plane or the image plane, S-FFT algorithm can not resolve the diffraction's inverse calculation question in the near field or the near image plane.

5 Application of Collins formula's inverse calculation in wavefront reconstruction

If CCD screen is regarded as an output plane after object light goes through an $ABCD$ system, then the objective optical wave field can be reconstructed by using inverse calculation of Collins' formula. And then it will become a specific example based on the wavefront reconstruction of inverse calculation of Fresnel diffraction integral when $A = 1$, $B = d$, $C = 0$, $D = 1$.

To certify the accuracy of theoretical results mentioned above, we have done experiment with YAG laser of wavelength 532 nm. Figure 1 is the illustration of the experiment. In Figure 1, the plane wave propagates along

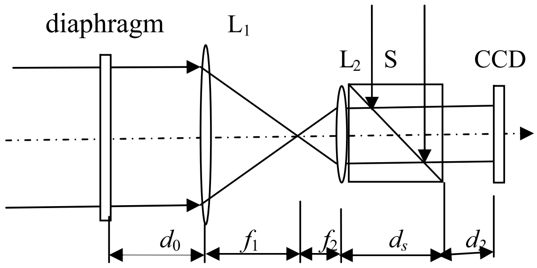


Fig. 1. Illustration of the experiment.

z-axis forward. This beam first illuminates a diaphragm and forms the optical wave field of incidence plane. The diaphragm is a transparent hole carved with a Chinese word "龙" (which means dragon) with the width 10 mm. Then the beam reaches the lens L_1 with focal length $f = 680$ mm after passing through a diffraction distance d_0. After that it reaches the lens L_2 with focal length $f = 150$ mm after passing through distance d_1. It then enters a beam splitter S, whose width is d_s and the refractive index is $n = 1.5$. The beam finally reaches CCD screen after passing through another distance d_2. In the experiment, $d_1 = f_1 + f_2$, and a plane beam is introduced above S as reference beam. The size of CCD is 3.3 mm × 3.3 mm, and pixels are 1024 × 1024.

The diaphragm plane and CCD plane are regarded as the incident plane and emergent plane of $ABCD$ system respectively, so the elements of matrix of $ABCD$ system can be described by the following equation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_s/n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & f_1+f_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -f_2/f_1 & f_1+f_2-(d_s/n+d)\, f_1/f_2 - d_0 f_2/f_1 \\ 0 & -f_1/f_2 \end{bmatrix}. \tag{15}$$

As equation $|A| = 1/|D|$ is satisfied, when inequation (14) take an equal mark, we can obtain:

$$\Delta L^2 = |AB\lambda N|\,. \tag{16}$$

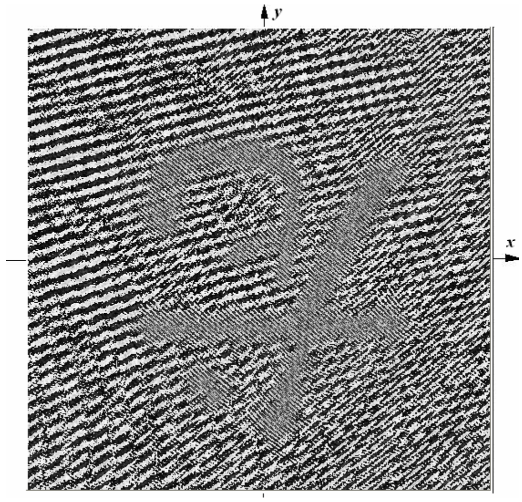
That is to say, when $\Delta L = \sqrt{|AB\lambda N|}$ is selected, the optical wave field on the incidence plane obtained by inverse calculation meets sampling theorem approximately. When the width of CCD window $\Delta L = \sqrt{|AB\lambda N|} = 3.3$ mm, then we have:

$$B = \frac{3.3 \times 3.3}{|150/680 \times 0.000532 \times 1024|} \approx 90.6 \text{ (mm)}.$$

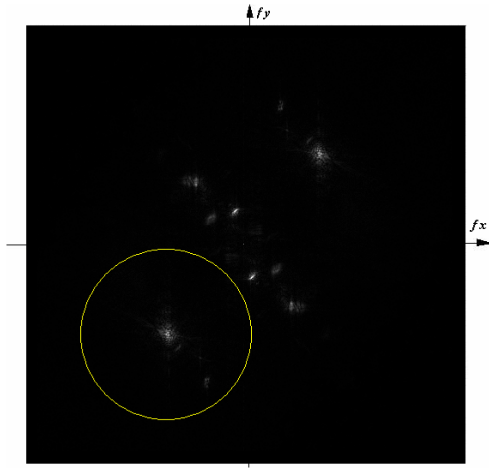
The width of the reconstructed object plane can be obtained by equation (10):

$$\Delta L_0 = \frac{\lambda |B| N}{\Delta L} = \frac{0.000532 \times 90.5 \times 1024}{3.3} \approx 15 \text{ mm}.$$

It is obvious that the reconstructed object plane which is obtained by using above parameters can completely contain original object plane's diaphragm. In addition, according to equation (15), the optical system parameters



(a) Gray level distribution of the subtracted image (1024 × 1024)



(b) Frequency spectrum intensity of the subtracted image (512 × 512)

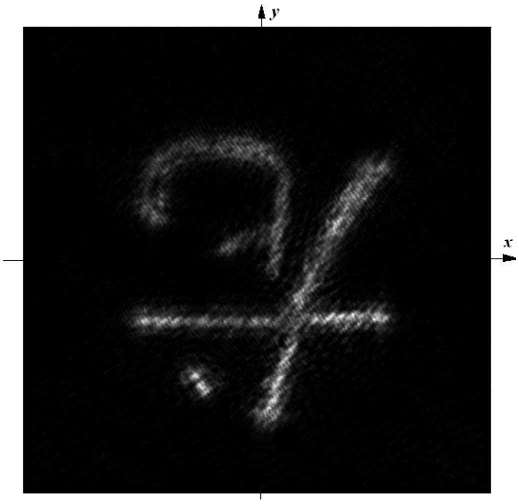
Fig. 2. Gray level distribution and the frequency spectrum intensity of the subtracted image which detected by CCD before and after introducing a phase-shift.

A, C and D are only related to the focal length of lens, and parameter B is determined by the following equation:

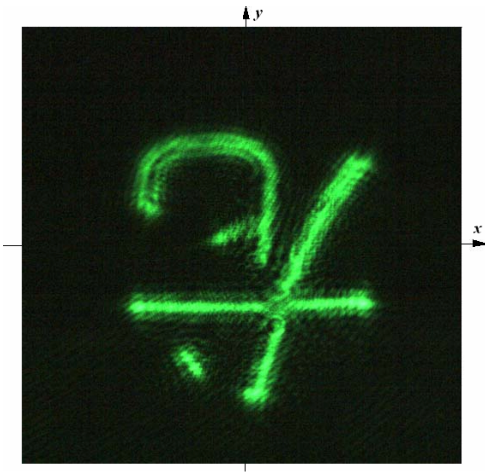
$$B = f_1 + f_2 - (d_s/n + d_2)\, f_1/f_2 - d_0 f_2/f_1. \tag{17}$$

Therefore, when d_s, n, f_1, f_2 are already defined, we can obtain the same system parameters A, B, C and D when different d_2, d_0 are selected. And it provides a proof to the proper selection for the location of object and CCD according to the experimental conditions. For example, supposing $d_0 = 260$ mm, we can obtain $d_2 \approx 97$ mm. The above results are certified by following experiment.

In the experimental research, a phase-shift (non integral multiples of 2π) is introduced in the reference light to eliminate the influence of zero-order diffraction light effectively. And then the frequency spectrum of object light is obtained by doing discrete Fourier transform on the

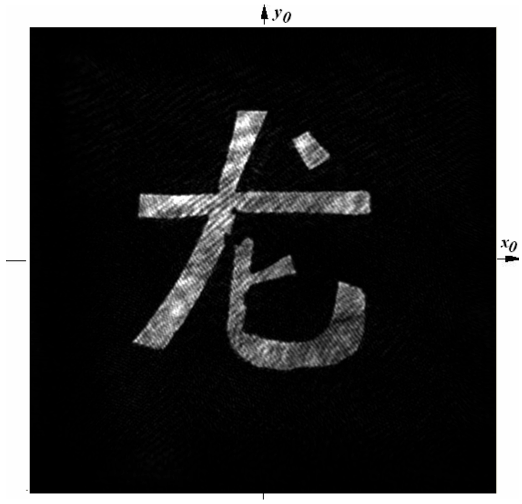


(a) Diffraction image obtained by calculation (3.3 mm×3.3 mm) inverse

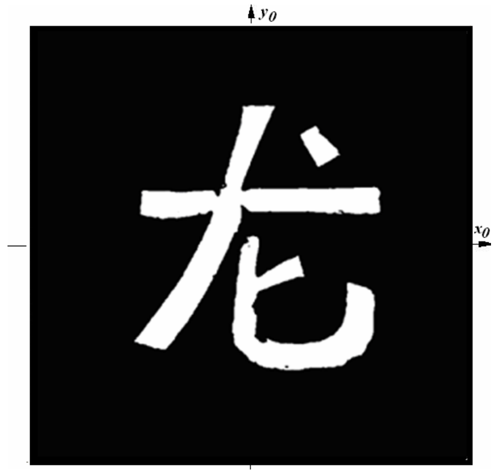


(b) Diffraction image detected by CCD (3.3 mm×3.3 mm)

Fig. 3. Comparison between the diffraction image obtained by inverse calculation on object light frequency spectrum and the actual measured image.



(a) Reconstructed object plane by inverse calculation (15 mm × 15 mm)



(b) Actual object plane (15 mm × 15 mm)

Fig. 4. Comparison between Collins formula and S-FFT calculation of its inverse calculation and experiment measurement.

difference image. The difference image is obtained by subtracting one intensity interference pattern from the other with phase-shifting of reference wave [6]. The gray level distribution and the frequency spectrum intensity of the subtracted image are shown as Figures 2a and 2b (passing region of the filter is shown with tint ring in Fig. 2b).

Shifting the frequency spectrum of object light obtained in Figure 2b to the center of frequency plane, and then using zero-filling operation to form a new object light frequency spectrum, the sampling points of which are 1024 × 1024, we can get the object light diffraction image obtained by doing inverse Fourier transform on the frequency spectrum as shown in Figure 3a. And the image used in the experiment to cover the reference light is recorded by CCD and is shown in Figure 3b. Obviously, theoretical calculation agrees well with experimental measurement.

The optical wave field data which correspond to Figure 3a are applied to inverse calculation equation (11). The normalized intensity distribution of the optical wave field on the object plane (0∼255) is shown as Figure 4a. Figure 4b is a diaphragm's projective image. It is obvious that there is no difference between the theoretically reconstructed image and the actual image virtually. But the light intensity which inner the transparent hole of the reconstructed image is uneven, and the border is relatively diffused. The main reason of the uneven light intensity is that the illuminating light is not uniform and there is a failure to sufficiently filter zero-order diffraction light and frequency spectrum of conjugate light in the frequency spectrum plane (Fig. 2b). But the diffuse boundary is due to the loss of high frequency when object frequency spectrum is obtained from Figure 2b. If the uniform degree of the illuminate light is improved, and three-time phase-shift ($\pi/2$, π, $3\pi/2$) is introduced in the reference light [1,5], a better reconstructed quality of the optical wave field on the plane of object can be obtained from object light frequency spectrum, which is from four interference images before and after introducing a phase-shift. However, the experiment has already given a certification to the discussion in this paper.

Certainly, the comparison mentioned above just provides a feasibility for the reconstruction of amplitude. Actually, phase reconstruction of inverse calculation is fairly precise. Calculation results indicate that the maximal change of phase is less than 10^{-12} radian in the transparent hole, so it can be regarded as a plane wave. Therefore, according to the calculated conditions derived from this paper, the inverse calculation equation (11) can reconstruct the incidence plane optical wave field accurately. And it is completely feasible to use S-FFT algorithm of inverse calculation of Collins' formula in digital holography.

6 Conclusions

In short, Collins' formula and S-FFT algorithm of its inverse calculation are studied in this paper. On the basis of sampling theorem, the condition equations that are used to calculate amplitude and phase of optical wave fields are deduced. Moreover, an application research example of digital holography based on the studied results is given. And it provides a useful reference for the calculation of Collins' formula and the study of digital holography.

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